SEMICROSSED PRODUCTS OF SIMPLE C*-ALGEBRAS.

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Abstract. Let $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ be C*-dynamical systems and assume that $\mathcal{A}$ is a separable simple C*-algebra and that α and β are $*$-automorphisms. Then the semicrossed products $\mathcal{A} \times_\alpha \mathbb{Z}^+$ and $\mathcal{B} \times_\beta \mathbb{Z}^+$ are isometrically isomorphic if and only if the dynamical systems $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ are outer conjugate.

1. Introduction

The main objective of this paper is the classification of semicrossed products of separable simple C*-algebras by an automorphism, up to isometric isomorphism. It is easily seen (and well-known) that outer conjugacy between automorphisms of arbitrary C* algebras is a sufficient condition for the existence of an isometric isomorphism between the associated semicrossed products. In this paper we show that for separable simple C*-algebras, this is also a necessary condition. This follows from the following general result: if α and β are automorphisms of arbitrary C*-algebras $\mathcal{A}$ and $\mathcal{B}$, then the presence of an isometric isomorphism from $\mathcal{A} \times_\alpha \mathbb{Z}^+$ onto $\mathcal{B} \times_\beta \mathbb{Z}^+$ implies the existence of a C*-isomorphism $\gamma : \mathcal{A} \to \mathcal{B}$ so that $\alpha \circ \gamma^{-1} \circ \beta^{-1} \circ \gamma$ is universally weakly inner with respect to irreducible representations. The result for simple C*-algebras follows then from a remarkable result of Kishimoto [**16**] which shows that for a separable simple C*-algebra, all universally weakly inner automorphisms are actually inner.

Let $(\mathcal{A}, \alpha)$ be a (discrete) C*-dynamical system, i.e., a C*-algebra $\mathcal{A}$ together with a $*$-endomorphism α of $\mathcal{A}$. Motivated by a construction of Arveson [**3**], Peters [**24**] introduced the concept of the semicrossed product $\mathcal{A} \times_\alpha \mathbb{Z}^+$. This is the universal operator algebra for contractive covariant representations of this system.

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In the commutative case, the semicrossed product is an algebra $\mathrm{C}_0(\mathcal{X}) \times_\sigma \mathbb{Z}^+$ determined by a dynamical system $(\mathcal{X}, \sigma)$ given by a proper continuous map σ acting on a locally compact Hausdorff space X. Under the assumption that the topological spaces are compact and the maps are aperiodic, Peters [**24**] showed that two such semicrossed products are isomorphic as algebras if and only if the corresponding dynamical systems are conjugate, thus extending an earlier classification scheme of Arveson [**3**] and Arveson and Josephson [**4**]. In spite of the subsequent interest in semicrossed products and their variants [**2, 5, 6, 13, 17, 18, 20, 23, 26, 27**], the problem of classifying semicrossed products of the form $\mathrm{C}_0(\mathcal{X}) \times_\sigma \mathbb{Z}^+$ remained open in the generality introduced by Peters in [**24**] until our recent paper [**8**], which established that the Arveson-Josephson-Peters classification scheme holds with no restrictions on either $\mathcal{X}$ or σ. It was our desire to apply the techniques of [**8**] and [**9**] to more general settings that motivated the research of the present paper.

The present paper provides for the first time a classification scheme for semicrossed products which is valid for a broad class of C*-algebras, without posing any restrictions on the automorphisms involved. Our result complements a similar result of Muhly and Solel [**21**, Theorem 4.1] regarding semicrossed products with automorphisms having full Connes spectrum. In Theorem 4.2, we give an alternative proof of their result using representation theory. Both results seem to indicate that outer conjugacy is a complete invariant for isometric isomorphisms between arbitrary semicrossed products. They also suggest the problem of establishing the validity of the conclusion under the weaker requirement of an *algebraic* isomorphism instead of an isometric isomorphism.

2. Preliminaries

Let $\mathcal{A}$ be a C*-algebra and α an endomorphism of $\mathcal{A}$. The *skew polynomial algebra* $P(\mathcal{A}, \alpha)$ consists of all polynomials of the form $\sum_n U_\alpha^n A_n$, $A_n \in \mathcal{A}$, where the multiplication of the "coefficients" $A \in \mathcal{A}$ with the "variable" U_α obeys the rule

$$AU_\alpha = U_\alpha \alpha(A)$$

Equip $P(\mathcal{A}, \alpha)$ with the l^1-norm

$$\Big\| \sum_n U_\alpha^n A_n \Big\|_1 \equiv \sum_n \|A_n\|$$

and let $l^1(\mathcal{A}, \alpha)$ be the completion of $P(\mathcal{A}, \alpha)$ with respect to $\| \,.\, \|_1$.

An *(isometric) covariant representation* (π, V) of $(\mathcal{A}, \alpha)$ consists of a C*-representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ and an isometry V

on $\mathcal{H}$ so that $\pi(A)V = V\pi(\alpha(A))$, for all $A \in \mathcal{A}$. Each covariant representation induces a representation $\pi \times V$ in an obvious way.

Definition 2.1. For $P \in l^1(\mathcal{A}, \alpha)$ let

$$\|P\| := \sup\left\{\|(\pi \times V)(P)\| : \pi \times V \quad \text{is covariant}\right\}$$

where $\pi \times V$ runs over all isometric covariant representations of $(\mathcal{A}, \alpha)$. The *semicrossed product* $\mathcal{A} \times_\alpha \mathbb{Z}^+$ of $\mathcal{A}$ by α is the completion of $l^1(\mathcal{A}, \alpha)$ with respect to this norm.

For an alternative description of $\mathcal{A} \times_\alpha \mathbb{Z}^+$, one may start by obtaining a universal covariant representation $(\pi \times V)$ of $(\mathcal{A}, \alpha)$ and then define $\mathcal{A} \times_\alpha \mathbb{Z}^+$ to be the non-sefadjoint operator algebra generated $\pi(\mathcal{A})$ and V. The two constructions produce isomorphic algebras. For each covariant representation (π, V) of $(\mathcal{A}, \alpha)$, the representation $\pi \times V$ extends uniquely to a contractive representation of $\mathcal{A} \times_\alpha \mathbb{Z}^+$, which will also be denoted as $\pi \times V$.

In this paper, we will exclusively work with *invertible* C*-dynamical systems, i.e., the endomorphism will actually be an automorphism. Therefore we now drop the adjective "invertible", and by C*-dynamical system we will mean an invertible one. Given an (invertible) dynamical system $(\mathcal{A}, \alpha)$, the unitary covariant representations for $(\mathcal{A}, \alpha)$ suffice to capture the norm for $\mathcal{A} \times_\alpha \mathbb{Z}^+$. Therefore, $\mathcal{A} \times_\alpha \mathbb{Z}^+$ is a natural nonselfadjoint subalgebra of the crossed product C*-algebra $\mathcal{A} \times_\alpha \mathbb{Z}$.

Definition 2.2. Two C*-dynamical systems $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ are said to be *outer conjugate* if there exists a C*-isomorphism $\gamma : \mathcal{A} \to \mathcal{B}$ and a unitary $W \in M(\mathcal{A})$, the multiplier algebra of $\mathcal{A}$, so that

$$\alpha = \mathrm{ad}_W \, \gamma^{-1} \circ \beta \circ \gamma.$$

The main issue in this paper is the classification of semicrossed products up to isometric isomorphism. The following elementary result shows that the outer conjugacy of automorphisms provides a sufficient condition for the existence of such an isomorphism.

Proposition 2.3. *If the dynamical systems $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ are outer conjugate, then the semicrossed products $\mathcal{A} \times_\alpha \mathbb{Z}^+$ and $\mathcal{B} \times_\beta \mathbb{Z}^+$ are isometrically isomorphic.*

Proof. Without loss of generality assume that $\mathcal{A} = \mathcal{B}$ and $\gamma = \mathrm{id}$. Let $W \in M(\mathcal{A})$ so that $\alpha(A) = W\beta(A)W^*$. Observe that β has a unique extension to an automorphism $\bar{\beta}$ of $M(\mathcal{A})$ such that

$$\bar{\beta}(M)\beta(A) = \beta(MA) \quad \text{for all} \quad A \in \mathcal{A} \text{ and } M \in M(\mathcal{A}),$$

namely, $\bar{\beta}(M)A = \beta(M\beta^{-1}(A))$. Therefore $\mathcal{A} \times_\beta \mathbb{Z}$ is naturally a subalgebra of $M(\mathcal{A}) \times_{\bar{\beta}} \mathbb{Z}$ generated by $\mathcal{A}$ and the universal unitary U_β satisfying $U_\beta M U_\beta^* = \beta(M)$ for $M \in M(\mathcal{A})$.

Now notice that ad_{WU_β} implements $\bar{\alpha}$, the extension of α to $M(\mathcal{A})$ because it is an automorphism which acts as α on $\mathcal{A}$. Whence it carries $M(\mathcal{A})$ to itself, and is the unique extension of α to $M(\mathcal{A})$. Therefore $\mathrm{C}^*(\mathcal{A}, WU_\beta)$ determines a representation σ of the C*-algebra crossed product $\mathcal{A} \times_\alpha \mathbb{Z}$ given by $\sigma|_{\mathcal{A}} = \mathrm{id}$ and $\sigma(U_\alpha) = U_\beta W$. Since this representation is faithful on $\mathcal{A}$, it follows from the gauge invariance uniqueness theorem [**15**, Theorem 6.4] that this is a faithful representation of the crossed product.

Next observe that $\mathrm{C}^*(\mathcal{A}, WU_\beta) = \mathcal{A} \times_\beta \mathbb{Z}$. The point is that

$$\begin{aligned} A(U_\beta W)^n &= A(U_\beta W U_\beta^*)(U_\beta^2 W U_\beta^{*2}) \cdots (U_\beta^n W U_\beta^{*n}) U_\beta^n \\ &= A\bar{\beta}(W)\bar{\beta}^2(W) \cdots \bar{\beta}^n(W) U_\beta^n \\ &= B U_\beta^n \end{aligned}$$

where $B = A\bar{\beta}(W)\bar{\beta}^2(W)\cdots\bar{\beta}^n(W)$ belongs to $\mathcal{A}$. It now follows that $\mathcal{A}(U_\beta W)^n = \mathcal{A}U_\beta^n$. Hence $\mathrm{C}^*(\mathcal{A}, WU_\beta) = \mathcal{A} \times_\beta \mathbb{Z}$. Moreover, the nonself-adjoint subalgebra $\mathcal{A} \times_\beta \mathbb{Z}^+$ generated by $\mathcal{A}$ and $\mathcal{A}U_\beta$ coincides with the algebra generated by $\mathcal{A}$ and $\mathcal{A}U_\beta W$. But this latter algebra is canonically identified with $\mathcal{A} \times_\alpha \mathbb{Z}^+$ via the identification of $\mathrm{C}^*(\mathcal{A}, WU_\beta)$ with $\mathcal{A} \times_\alpha \mathbb{Z}$. ∎

Note that $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ are outer conjugate if and only if there exists a C*-isomorphism $\gamma : \mathcal{A} \to \mathcal{B}$ so that the automorphism $a \circ \gamma^{-1} \circ \beta^{-1} \circ \gamma$ is inner. A notion weaker than that of outer congucacy arises from the concept of a universally weakly inner automorphism. We say an automorphism a of $\mathcal{A}$ is *universally weakly inner* with respect to irreducible (resp. faithful) representations, if for any irreducible (resp. faithful) representation π of $\mathcal{A}$ on $\mathcal{H}$, there exists a unitary $W \in \pi(\mathcal{A})''$ so that $\pi(\alpha(A)) = W^*\pi(A)W$, $A \in \mathcal{A}$. The concept of a universally weakly inner automorphism with respect to faithful representations (or π-inner automorphism) was introduced by Kadison and Ringrose [**14**] and has been studied by various authors [**10, 19**]. Here we will be making use of universally weakly inner automorphisms with respect to *irreducible* representations. A direct integral decomposition argument shows that the two concepts coincide for type I C*-algebras. Kishimoto [**16**] has shown that for a separable simple C*-algebra all universally weakly inner automorphisms with respect to irreducible representations are actually inner. Therefore the two concepts coincide there as well.

We also need to recall several facts for the various concepts of spectrum from C*-algebra theory. Let $(\mathcal{A}, \alpha)$ be a dynamical system and let $\operatorname{Prim} A$ be the primitive ideal space of $\mathcal{A}$ equipped with the hull-kernel topology. For each cardinal i, we fix a Hilbert space $\mathcal{H}_i$ with dimension i. Then, $\operatorname{irred}(\mathcal{A}, \mathcal{H}_i)$ is the collection of all irreducible representations on $\mathcal{H}_i$ and $\operatorname{irred}\mathcal{A} \equiv \cup_i \operatorname{irred}(\mathcal{A}, \mathcal{H}_i)$. If $\rho \in \operatorname{irred}\mathcal{A}$, then $[\rho]$ denotes its equivalence class, with respect to unitary equivalence $\simeq$ between representations on the same Hilbert space. Let the *spectrum* of $\mathcal{A}$ be

$$\hat{\mathcal{A}} \equiv \big\{[\rho] \mid \rho \in \operatorname{irred}\mathcal{A}\big\}$$

and consider the canonical map

$$\theta : \hat{\mathcal{A}} \longrightarrow \operatorname{Prim}\mathcal{A} : [\rho] \longrightarrow \ker \rho.$$

In what follows we always consider $\hat{\mathcal{A}}$ equiped with the smallest topology that makes θ continuous. For any C*-isomorphism $\gamma : \mathcal{A} \to \mathcal{B}$, we define a map $\hat{\gamma} : \hat{\mathcal{A}} \to \hat{\mathcal{B}}$ between the corresponding spectra by the formula $\hat{\gamma}([\rho]) = [\rho \circ \gamma]$.

The following result is straightforward.

Lemma 2.4. *Let $\mathcal{A}$ be a C*-algebra and let $\mathcal{X} \subseteq \hat{\mathcal{A}}$ have empty interior. Then,*

$$\bigcap_{x \in \hat{\mathcal{A}} \setminus \mathcal{X}} \theta(x) = \{0\}.$$

Another notion of spectrum is the Connes spectrum. We do not give the precise definition but instead state the fact [**22**, Theorem 10.4] that for a separable C*-algebra, $(\mathcal{A}, \alpha)$ has full Connes spectrum if and only if there is a dense α-invariant subset $\Delta_\alpha \subseteq \hat{\mathcal{A}}$ on which $\hat{\alpha}$ is freely acting. This is equivalent to the fact that the periodic points of $\hat{\alpha}$ with period n has no interior for any $n \geq 1$.

3. The main result

We begin this section with a general result about isometric isomorphisms between arbitrary operator algebras which is well-known.

Proposition 3.1. *Let $\phi : \mathfrak{A} \to \mathfrak{B}$ be an isometric isomorphism between operator algebras. Then $\phi(\mathfrak{A} \cap \mathfrak{A}^*) = \mathfrak{B} \cap \mathfrak{B}^*$ and $\phi|_{\mathfrak{A} \cap \mathfrak{A}^*}$ is a C*-isomorphism.*

Proof. The unitary operators in $\mathfrak{A} \cap \mathfrak{A}^*$ are characterized as the norm 1 elements $A \in \mathcal{A}$ so that $A^{-1} \in \mathfrak{A}$ and $\|A^{-1}\| = 1$. From this it follows that ϕ maps the unitary group of $\mathfrak{A} \cap \mathfrak{A}^*$ onto the unitary group of $\mathfrak{B} \cap \mathfrak{B}^*$, and this proves the first assertion. The second follows from

the fact that ϕ preserves inverses of unitaries in $\mathfrak{A} \cap \mathfrak{A}^*$ and hence adjoints. ∎

In order to prove the main result, we need to use representation theory. In light of Proposition 3.1, it suffices to consider representations that preserve the diagonal. Hence, if $\mathfrak{A}$ is an operator algebra, then $\mathrm{rep}(\mathfrak{A}, \mathcal{H})$ will denote the collection of all contractive representations of $\mathfrak{A}$ on $\mathcal{H}$ whose restriction on the diagonal $\mathfrak{A} \cap \mathfrak{A}^*$ is a $*$-homomorphism.

Definition 3.2. If $\mathfrak{A}$ is an operator algebra and $\mathcal{H}$ a Hilbert space, then $\mathrm{rep}_2(\mathfrak{A}, \mathcal{H})$ will denote the collection of all (contractive) representations $\rho \in \mathrm{rep}(\mathfrak{A}, \mathcal{H} \oplus \mathcal{H})$ of the form

$$\rho(A) = \begin{bmatrix} \rho_1(A) & \rho_3(A) \\ 0 & \rho_2(A) \end{bmatrix} \quad \text{for all} \quad A \in \mathfrak{A},$$

so that $\rho_i|_{\mathfrak{A}\cap\mathfrak{A}^*} \in \mathrm{irred}(\mathfrak{A} \cap \mathfrak{A}^*, \mathcal{H})$ for $i = 1, 2$ and $\rho_3(\mathfrak{A}) \neq \{0\}$.

Lemma 3.3. *Let $(\mathcal{A}, \alpha)$ be a C*-dynamical system and let*

$$\rho \in \mathrm{rep}_2(\mathcal{A} \times_\alpha \mathbb{Z}^+, \mathcal{H}).$$

Then, $\rho_1|_{\mathcal{A}} \simeq \rho_2|_{\mathcal{A}} \circ \alpha$.

Proof. First note that ρ is a $*$-representation on the diagonal $\mathcal{A}$. Hence for each $A \in \mathcal{A}$, $\rho(A)$ necessarily has its $(1, 2)$-entry equal to zero, i.e., $\rho(\mathcal{A})$ is in diagonal form.

Let$\{E_j\}_j$ be an approximate unit for $\mathcal{A}$; and let $X, Y, Z \in \mathcal{H}$ so that $\{\rho(U_\alpha E_j)\}_j$ converges weakly to $\left[\begin{smallmatrix} X & Y \\ 0 & Z \end{smallmatrix}\right]$. *We claim that* $Y \neq 0$. Indeed, otherwise the equality

$$\rho(U_\alpha A) = \lim_j \rho(U_\alpha E_j A) = \lim_j \rho(U_\alpha E_j)\rho(A) \quad \text{for all} \quad A \in \mathcal{A}$$

would imply that $\rho(U_\alpha \mathcal{A})$ is diagonal; and therefore $\rho(\mathcal{A} \times_\alpha \mathbb{Z}^+)$ is in diagonal form, contradicting the requirement $\rho_3(\mathcal{A} \times_\alpha \mathbb{Z}^+) \neq \{0\}$.

Now notice that for any $A \in \mathcal{A}$, we have

$$\begin{aligned} \rho(A) \lim_j \rho(U_\alpha E_j) &= \lim_j \rho(U_\alpha \alpha(A) E_j) \\ &= \lim_j (U_\alpha E_j)\rho(\alpha(A)). \end{aligned}$$

Hence in matricial form,

$$\begin{bmatrix} \rho_1(A) & 0 \\ 0 & \rho_2(A) \end{bmatrix} \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \begin{bmatrix} \rho_1(\alpha(A)) & 0 \\ 0 & \rho_2(\alpha(A)) \end{bmatrix}.$$

By multiplying and comparing $(1, 2)$-entries, we obtain

$$\rho_1(A)Y = Y\rho_2(\alpha(A)).$$

Since $Y \neq 0$ and $\rho_1|_{\mathcal{A}}$ and $\rho_2|_{\mathcal{A}}$ are irreducible, this implies that $\rho_1|_{\mathcal{A}} \simeq \rho_2|_{\mathcal{A}} \circ \alpha$, as desired. ∎

Lemma 3.4. *Let $(\mathcal{A}, \alpha)$ be a C*-dynamical system, and let σ belong to* $\mathrm{irred}(\mathcal{A}, \mathcal{H})$*. Then there exists a representation $\rho \in \mathrm{rep}_2(\mathcal{A} \times_\alpha \mathbb{Z}^+, \mathcal{H})$ so that $\rho_2|_{\mathcal{A}} = \sigma$.*

Proof. Let

$$\rho(A) = \begin{bmatrix} \sigma \circ \alpha(A) & 0 \\ 0 & \sigma(A) \end{bmatrix} \quad \text{and} \quad \rho(U_\alpha A) = \begin{bmatrix} 0 & \sigma(A) \\ 0 & 0 \end{bmatrix}.$$

It is easily verified that $\left(\rho|_{\mathcal{A}}, \left[\begin{smallmatrix} 0 & I \\ 0 & 0 \end{smallmatrix}\right]\right)$ is a covariant representation and so ρ extends to a representation of $\mathcal{A} \times_\alpha \mathbb{Z}^+$. ∎

Note that the representation ρ in Lemma 3.4 satisfies $\rho_1|_{\mathcal{A}} = \sigma \circ \alpha$. This is not just an artifact of our construction. By Lemma 3.3, any representation in $\rho \in \mathrm{rep}_2(\mathcal{A} \times_\alpha \mathbb{Z}^+, \mathcal{H})$ will satisfy that property, provided that $\rho_2|_{\mathcal{A}} = \sigma$.

The following general result relates the classification problem for semicrossed products to the study of universally weakly inner automorphisms for C*-algebras.

Theorem 3.5. *Let $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ be C*-dynamical systems, and assume that the semicrossed products $\mathcal{A} \times_\alpha \mathbb{Z}^+$ and $\mathcal{B} \times_\beta \mathbb{Z}^+$ are isometrically isomorphic. Then there exists a C*-isomorphism $\gamma : \mathcal{A} \to \mathcal{B}$ so that $\alpha \circ \gamma^{-1} \circ \beta^{-1} \circ \gamma$ is universally weakly inner with respect to irreducible representations.*

Proof. Assume that there exists an isometric isomorphism

$$\gamma : \mathcal{A} \times_\alpha \mathbb{Z}^+ \longrightarrow \mathcal{B} \times_\beta \mathbb{Z}^+.$$

By Proposition 3.1, $\gamma|_{\mathcal{A}}$ is a $*$-isomorphism onto $\mathcal{B}$, which we will also denote by γ; this is the promised isomorphism. Indeed γ establishes a correspondence

$$\mathrm{irred}(\mathcal{A}) \ni \sigma \longrightarrow \sigma \circ \gamma^{-1} \in \mathrm{irred}(\mathcal{B})$$

that preserves equivalence classes. To show that $\alpha \circ \gamma^{-1} \circ \beta^{-1} \circ \gamma$ is universally weakly inner with respect to irreducible representations, it is enough to show that

$$\sigma \circ \alpha \circ \gamma^{-1} \simeq \sigma \circ \gamma^{-1} \circ \beta$$

for any $\sigma \in \mathrm{irred}(\mathcal{A})$.

By Lemma 3.4, there exists $\rho \in \mathrm{rep}_2(\mathcal{A} \times_\alpha \mathbb{Z}^+, \mathcal{H})$ so that $\rho_2|_{\mathcal{A}} = \sigma$. But then

$$\rho \circ \gamma^{-1} \in \mathrm{rep}_2(\mathcal{B} \times_\beta \mathbb{Z}^+, \mathcal{H}).$$

Hence, Lemma 3.2 implies that

$$(\rho \circ \gamma^{-1})_1|_{\mathcal{B}} \simeq (\rho \circ \gamma^{-1})_2|_{\mathcal{B}} \circ \beta$$

or, once again, $\sigma \circ \alpha \circ \gamma^{-1} \simeq \sigma \circ \gamma^{-1} \circ \beta$. ∎

We have arrived to the main result of the paper.

Theorem 3.6. *Let $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ be C*-dynamical systems and assume that $\mathcal{A}$ is a separable simple C*-algebra. The semicrossed products $\mathcal{A} \times_\alpha \mathbb{Z}^+$ and $\mathcal{B} \times_\beta \mathbb{Z}^+$ are isometrically isomorphic if and only if the dynamical systems $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ are outer conjugate.*

Proof. One direction follows from Proposition 2.3. Conversely assume that $\mathcal{A} \times_\alpha \mathbb{Z}^+$ and $\mathcal{B} \times_\beta \mathbb{Z}^+$ are isometrically isomorphic. Theorem 3.5 shows that there exists a C*-isomorphism $\gamma : \mathcal{A} \to \mathcal{B}$ so that $\alpha \circ \gamma^{-1} \circ \beta^{-1} \circ \gamma$ is universally weakly inner with respect to irreducible representations. The conclusion follows now from Kishimoto's result [**16**, Corollary 2.3]. ∎

4. Representations and dynamics on the spectrum

As we mentioned in the introduction, an earlier result of Muhly and Solel [**21**, Theorem 4.1] implies the validity of Theorem 3.6 for arbitrary C*-algebras, provided that the automorphisms α and β have full Connes spectrum. In what follows we present an alternative proof of that result of Muhly and Solel, based on the ideas developed in this paper.

Let $\hat{\Phi}_\alpha$ denote the canonical expectation from $\mathcal{A} \times_\alpha \mathbb{Z}$ onto $\mathcal{A}$ and let Φ_α denote its restriction on the semicrossed product $\mathcal{A} \times_\alpha \mathbb{Z}^+$. The following is the key step in their proof.

Lemma 4.1. *Let $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ be separable C*-dynamical systems and let $\gamma : \mathcal{A} \times_\alpha \mathbb{Z}^+ \to \mathcal{B} \times_\beta \mathbb{Z}^+$ be an isometric isomorphism. If α has full Connes spectrum, then $\gamma(\ker \Phi_\alpha) = \ker \Phi_\beta$.*

Proof. Let $\Delta_\alpha = \{\sigma_j \mid j \in \mathbb{J}\}$ be the dense α-invariant set of aperiodic points described in the introduction. By Lemma 2.4, $\oplus_j \sigma_j$ is a faithful representation of $\mathcal{A}$ and so $(\oplus_j \sigma_j) \times_\alpha V$ defines a faithful representation of $\mathcal{A} \times_\alpha \mathbb{Z}$. Now notice that the compression on the main diagonal of $(\oplus_j \sigma_j) \times_\alpha V$ defines an expectation from $\mathcal{A} \times_\alpha \mathbb{Z}$ onto $\mathcal{A}$ which coincides with $\hat{\Phi}_\alpha$. Similarly, $(\oplus_j \sigma_j \circ \gamma^{-1}) \times_\beta V$ defines a faithful representation of $\mathcal{B} \times_\beta \mathbb{Z}$ and $\hat{\Phi}_\beta$ is the compression on the main diagonal.

Now $\gamma(\mathcal{A}) = \mathcal{B}$ and $\ker \Phi_\alpha = U_\alpha (\mathcal{A} \times_\alpha \mathbb{Z}^+)$. Hence it is enough to show that $\gamma(U_\alpha) \in \ker \Phi_\beta$, i.e., the diagonal entries of $\gamma(U_\alpha)$ in the representation $(\oplus_j \sigma_j \circ \gamma^{-1}) \times_\beta V$ of $\mathcal{B} \times_\beta \mathbb{Z}^+$ are equal to 0. To

verify this examine these entries in light of the covariance equation as in Lemma 3.3. ∎

The rest of the proof follows the same arguments as in [**21**].

Theorem 4.2 ([**21**]). *Let $(\mathcal{A},\alpha)$ and $(\mathcal{B},\beta)$ be separable C*-dynamical systems and assume that α has full Connes spectrum. The semicrossed products $\mathcal{A}\times_\alpha\mathbb{Z}^+$ and $\mathcal{B}\times_\beta\mathbb{Z}^+$ are isometrically isomorphic if and only if the dynamical systems $(\mathcal{A},\alpha)$ and $(\mathcal{B},\beta)$ are outer conjugate.*

Proof. Assume that there is an isometric isomorphism γ of $\mathcal{A}\times_\alpha\mathbb{Z}^+$ onto $\mathcal{B}\times_\beta\mathbb{Z}^+$. By the previous Lemma we have

$$\begin{aligned}\gamma(U_\alpha) &= BU_\beta + Y, \quad \text{where } B\in\mathcal{B} \text{ and } Y\in U_\beta^2(\mathcal{B}\times_\beta\mathbb{Z}^+),\\ \gamma^{-1}(U_\beta) &= AU_\alpha + Z, \quad \text{where } A\in\mathcal{A} \text{ and } Z\in U_\alpha^2(\mathcal{A}\times_\alpha\mathbb{Z}^+).\end{aligned}$$

Since both γ and γ^{-1} are isometries, $\|A\|,\|B\|\le 1$. Also,

$$U_\alpha = \gamma^{-1}(\gamma(U_\alpha)) = \gamma^{-1}(B)AU_\beta + \gamma^{-1}(B)Z + \gamma^{-1}(Y),$$

which by the uniqueness of the Fourier expansion implies that $\gamma^{-1}(B)A = I$. Since A and $\gamma^{-1}(B)$ are contractions, they must both be unitary. Hence B and $\gamma(A)$ are also unitary. Therefore

$$\begin{aligned}\hat{\Phi}_\beta(\gamma(U_\alpha)^*\gamma(U_\alpha)) &= \hat{\Phi}_\beta(I + U_\beta^*B^*Y + Y^*BU_\beta + Y^*Y)\\ &= I + \hat{\Phi}_\beta(Y^*Y)\end{aligned}$$

since

$$U_\beta^*B^*Y \in U_\beta^*\mathcal{B}U_\beta^2(\mathcal{B}\times_\beta\mathbb{Z}^+) \subseteq U_\beta(\mathcal{B}\times_\beta\mathbb{Z}^+) \subseteq \ker\Phi_\beta;$$

and likewise $Y^*BU_\beta \in (\ker\Phi_\beta)^* \subseteq \ker\hat{\Phi}_\beta$. So $\hat{\Phi}_\beta(Y^*Y)=0$; whence $Y=0$. Hence, $\gamma(U_\alpha)=BU_\beta$ which implies that $(\mathcal{A},\alpha)$ and $(\mathcal{B},\beta)$ are outer conjugate. ∎

We finish the paper with a result of independent interest that associates the fixed points of $\hat{\alpha}$ to a certain analytic structure in the space of representations $\operatorname{rep}(\mathcal{A}\times_\alpha\mathbb{Z}^+,\mathcal{H})$.

Definition 4.3. Let $\mathfrak{A}$ be an operator algebra. A map

$$\Pi : \mathbb{D} \equiv \{z\in\mathbb{C} \mid |z|<1\} \longrightarrow (\operatorname{rep}(\mathfrak{A},\mathcal{H}), \text{point–}\textsc{sot})$$

is called *analytic* if for each $A\in\mathfrak{A}$ and $x,y\in\mathcal{H}$ the map $z\to\langle\Pi(z)(A)x \mid y\rangle$, $z\in\mathbb{D}$, is analytic in the usual sense.

Assume now that $(\mathcal{A},\alpha)$ is a C*-dynamical system, and let

$$\Pi : \mathbb{D} \longrightarrow \operatorname{rep}(\mathcal{A}\times_\alpha\mathbb{Z}^+,\mathcal{H})$$

be an analytic map. Since $\Pi(z)$, $z \in \mathbb{D}$, is a $*$-homomorphism on the diagonal, we may write

$$\Pi(z) = \pi_z \times K_z, \quad z \in \mathbb{D},$$

where $\pi_z = \Pi(z)|_{\mathcal{A}}$ and $K_z = \Pi(z)(U_\alpha)$. We claim that the map $z \longrightarrow \pi_z$, $z \in \mathbb{D}$, is constant. Indeed, consider any selfadjoint operator $A \in \mathcal{A}$ and notice that the map

$$\mathbb{D} \ni z \longrightarrow \langle \pi_z(A)x, x\rangle, \quad x \in \mathcal{H},$$

is real analytic and therefore constant, which proves the claim.

Definition 4.4. Let $\mathfrak{A}$ be an operator algebra. An analytic map $\Pi : \mathbb{D} \to \mathrm{rep}(\mathfrak{A}, \mathcal{H})$ is said to be *irreducible on the diagonal* if for any $z \in \mathbb{D}$, the representation $\Pi(z)|_{\mathfrak{A}\cap\mathfrak{A}^*}$ is irreducible.

Proposition 4.5. *Let $(\mathcal{A}, \alpha)$ be a C^*-dynamical system, and σ belong to* $\mathrm{irred}(\mathcal{A}, \mathcal{H})$*. Then $\sigma \simeq \sigma\circ\alpha$ if and only if there exists a non-constant analytic and irreducible on the diagonal map*

$$\Pi : \mathbb{D} \longrightarrow \mathrm{rep}(\mathcal{A} \times_\alpha \mathbb{Z}^+, \mathcal{H})$$

so that

$$\Pi(z)|_{\mathcal{A}} = \sigma$$

for some $z \in \mathbb{D}$.

Proof. First assume that $\sigma \simeq \sigma \circ \alpha$ and let U be a unitary such that $\sigma(A)U = U(\sigma \circ \alpha)(A)$ for $A \in \mathcal{A}$. Then

$$\Pi(z) = \sigma \times zU \quad \text{for} \quad z \in \mathbb{D}$$

has the desired properties.

Conversely, assume that such a map Π exists. By the discussion above $\Pi(z)|\mathcal{A} = \sigma$ for all $z \in \mathbb{D}$. Since Π is not constant, there exists some $z \in \mathbb{D}$ so that $K := \Pi(z)(U_\alpha) \neq 0$. Therefore,

$$\begin{aligned}\sigma(A)K &= \Pi(z)(A)\Pi(z)(U_\alpha)\\ &= \Pi(z)\left(U_\alpha\alpha(A)\right)\\ &= K\sigma(\alpha(A)) \qquad \text{for all} \quad A \in \mathcal{A}.\end{aligned}$$

Since both σ and $\sigma \circ \alpha$ are irreducible and $K \neq 0$, we obtain that $\sigma \simeq \sigma \circ \alpha$. ∎

Remark 4.6. Note that if Π is as in the above Proposition, then the operator K in the proof is necessarilly a scalar multiple of the (unique) unitary operator U implementing the equivalence $\sigma \simeq \sigma \circ \alpha$. Therefore the range of Π is contained in

$$D_\sigma \equiv \{\sigma \times zU \mid z \in \overline{\mathbb{D}}\}.$$

In this fashion, we associate with each fixed point $\sigma \in \hat{\mathcal{A}}$, a unique *maximal analytic set* D_σ. Any representation of $\mathcal{A} \times_\alpha \mathbb{Z}^+$ not belonging to the union of the maximal analytic sets is associated with a non-fixed point of $\hat{\alpha}$.

References

[1] C. Akemann, G. Pedersen and J. Tomiyama, *Multipliers of C*-Algebras*, J. Funct. Anal. **13** (1973), 277–301.

[2] M. Alaimia and J. Peters, *Semicrossed products generated by two commuting automorphisms* J. Math. Anal. Appl. **285** (2003), 128–140.

[3] W. Arveson, *Operator algebras and measure preserving automorphisms*, Acta Math. **118**, (1967), 95–109.

[4] W. Arveson and K. Josephson, *Operator algebras and measure preserving automorphisms II*, J. Funct. Anal. **4** (1969), 100–134.

[5] D. Buske, *Hilbert modules over a class of semicrossed products*, Proc. Amer. Math. Soc. **129** (2001), 1721–1726.

[6] D. Buske and J. Peters, *Semicrossed products of the disk algebra: contractive representations and maximal ideals.*, Pacific J. Math. **185** (1998), 97–113.

[7] L. DeAlba and Peters, J. *Classification of semicrossed products of finite-dimensional C*-algebras*, Proc. Amer. Math. Soc. **95** (1985), 557–564.

[8] K. Davidson and E. Katsoulis *Isomorphisms between topological conjugacy algebras*, J. Reine Angew. Math. (Crelle), to appear.

[9] K. Davidson and E. Katsoulis *Operator algebras for multivariable dynamics*, preprint, 2007.

[10] G. Elliott, *Ideal preserving automorphisms of postliminary C*-algebras*, Proc. Amer. Math. Soc. **27**, (1971), 107–109.

[11] D. Hadwin and T. Hoover, *Operator algebras and the conjugacy of transformations*, J. Funct. Anal. **77** (1988), 112–122.

[12] T. Hoover, *Isomorphic operator algebras and conjugate inner functions*, Michigan Math. J. **39** (1992), 229–237.

[13] T. Hoover, J. Peters and W. Wogen, *Spectral properties of semicrossed products.* Houston J. Math. **19** (1993), 649–660.

[14] R. Kadison and Ringrose, J. *Derivations and automorphisms of operator algebras*, Comm. Math. Phys. **4** (1967), 32–63.

[15] T. Katsura, *On C*-algebras associated with C*-correspondences*, J. Funct. Anal. **217** (2004), 366–401.

[16] A. Kishimoto, *Outer automorphisms and reduced crossed products of simple C*-algebras*, Comm. Math. Phys. **81** (1981), 429–435.

[17] M. Lamoureux, *Nest representations and dynamical systems*, J. Funct. Anal. **114** (1993), 467–492.

[18] M. Lamoureux, *Ideals in some continuous nonselfadjoint crossed product algebras.* J. Funct. Anal. **142** (1996), 211–248.

[19] E.C. Lance, *Automorphisms of postliminal C*-algebras*, Pacific J. Math. **23**, (1967), 547–555.

[20] M. McAsey and P. Muhly, *Representations of nonselfadjoint crossed products*, Proc. London Math. Soc. (3) **47** (1983), 128–144.

[21] P. Muhly, B. Solel, *On the Morita equivalence of tensor algebras*, Proc. London Math. Soc. (3) **81** (2000), 113–168.

[22] D. Olesen and Pedersen, G. *Applications of the Connes spectrum to C*-dynamical systems III*, J. Funct. Anal. **45** (1982), 357–390.

[23] T. Ohwada and K. Saito, *Factorization in analytic crossed products*, J. Math. Soc. Japan **54** (2002), 21–33.

[24] J. Peters, *Semicrossed products of C*-algebras*, J. Funct. Anal. **59** (1984), 498–534.

[25] J. Peters, *The ideal structure of certain nonselfadjoint operator algebras*, Trans. Amer. Math. Soc. **305** (1988), 333–352.

[26] S. Power, *Classification of analytic crossed product algebras*, Bull. London Math. Soc. **24** (1992), 368–372.

[27] B. Solel, *The invariant subspace structure of nonselfadjoint crossed products*, Trans. Amer. Math. Soc. **279** (1983), 825–840.

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